

# Infinite sequences of $p$ -groups with fixed coclass

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## Abstract

Eick & Leedham-Green sketched a construction for infinite sequences of finite  $p$ -groups with fixed coclass. These infinite sequences have turned out to be very useful in the theory of finite  $p$ -groups. We exhibit a detailed description for the construction of the finite sequences and we determine presentations for the infinite sequences for the primes and coclasses  $(2, 1)$ ,  $(2, 2)$  and  $(3, 1)$ .

## 1 Introduction

Coclass theory has been initiated by Leedham-Green and Newman [10]: they suggested to classify and investigate finite  $p$ -groups using their coclass as primary invariant. Recall that the coclass of a group of order  $p^n$  and class  $c$  is defined as  $n - c$ . The suggestion to classify  $p$ -groups by coclass has led to a rich and interesting research project. The book by Leedham-Green and McKay [9] describes its state of the art up until 2002.

The graphs  $\mathcal{G}(p, r)$  visualize the finite  $p$ -groups of coclass  $r$  and they play a central role in coclass theory. A recent discovery is the existence of periodic patterns in the graphs. These have been conjectured by Newman & O'Brien [11] and a first proof has been given by du Sautoy [3]. Eick & Leedham-Green [7] exhibited a new proof which additionally yields a group theoretic construction for infinite sequences of finite  $p$ -groups with coclass  $r$  associated to the periodic patterns in  $\mathcal{G}(p, r)$ .

The infinite sequences have proved to be useful in the theory of finite  $p$ -groups. For example, [4] determines a formula for the orders of the automorphism groups of the groups in an infinite sequences of 2-groups. The result implies that among the finite 2-groups of coclass  $r$  there are at most finitely many counterexamples to the long-standing divisibility conjecture: ‘if  $G$  is a non-abelian finite  $p$ -group, then  $|G|$  divides  $|Aut(G)|$ ’. Further, [5, 6] investigate the Schur multipliers of the groups in an infinite sequence. [6] shows that these Schur multipliers can be described by a single parametrised presentation. It is conjectured that this type of behaviour extends to homology groups and cohomology rings in general; See also [2] for some details.

The explicit definition of the infinite sequences is a side-result in [7] and it is not easily extracted from that paper. Further, in many of the applications of the infinite sequences minor variations on their construction are needed. It is the aim of this note to provide a detailed description for the construction of the infinite sequences of finite  $p$ -groups of coclass  $r$  so that a range of minor variations is available as well. As example, we construct the infinite sequences for the graphs  $\mathcal{G}(2, 1)$ ,  $\mathcal{G}(3, 1)$  and  $\mathcal{G}(2, 2)$ .

## 2 Preliminaries

In this section we summarize some of the well known major features of coclass theory. For background and proofs we refer to the book by Leedham-Green and McKay [9]. Throughout, for any group  $G$  we denote with  $G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$  the lower central series of  $G$  and we write  $G_{/i} = G/\gamma_i(G)$  for its quotients.

### 2.1 Infinite pro- $p$ -groups of finite coclass

Let  $S$  be a pro- $p$ -group with finite commutator quotient. Then every lower central series quotient  $S_{/i}$  is a finite  $p$ -group. We use this to define the coclass of  $S$  as  $\lim_{i \rightarrow \infty} cc(S_{/i})$ . Thus if  $S$  is a pro- $p$ -group of coclass  $r$ , then there exists an integer  $u_1$  such that  $cc(S_{/i}) = r$  and  $[\gamma_i(S) : \gamma_{i+1}(S)] = p$  holds for every  $i \geq u_1$ .

A general structure theory for the infinite pro- $p$ -groups of coclass  $r$  is available. A deep result of this theory asserts that every such group  $S$  is solvable. More precisely, every such group  $S$  has a torsion-free abelian normal subgroup of finite index. This implies that there exists an integer  $u_2$  such that  $\gamma_{u_2}(S)$  is torsion-free abelian and hence a direct product of finitely many copies of the  $p$ -adic integers  $\mathbb{Z}_p$ . The number of copies  $d$  is an invariant for  $S$ , called its *dimension*.

A fundamental theorem from coclass theory shows that there are only finitely many isomorphism types of infinite pro- $p$ -groups of coclass  $r$ . Further, every pair  $S$  and  $\bar{S}$  of non-isomorphic infinite pro- $p$ -groups of coclass  $r$  can be distinguished by finite quotients: there exists an integer  $u_3$  such that  $S_{/u_3} \not\cong \bar{S}_{/u_3}$  holds.

We define the *primary root* of an infinite pro- $p$ -group  $S$  of coclass  $r$  and dimension  $d$  as the smallest integer  $u$  such that the quotient  $S_{/u}$  is a finite  $p$ -group of coclass  $r$ , the subgroup  $\gamma_u(S)$  is torsion-free abelian and  $S_{/u}$  is not isomorphic to  $\bar{S}_{/u}$  for any infinite pro- $p$ -group  $\bar{S}$  of coclass  $r$  with  $\bar{S} \not\cong S$ .

### 2.2 Coclass graphs

The *coclass graph*  $\mathcal{G}(p, r)$  is a tool which is used to visualize the  $p$ -groups of coclass  $r$ . Its vertices correspond one-to-one to the isomorphism types of finite  $p$ -groups of coclass  $r$ . Two vertices  $G$  and  $H$  are connected by an edge (sometimes directed  $G \rightarrow H$ ) if  $H/\gamma(H) \cong G$ , where  $\gamma(H)$  is the last non-trivial term of the lower central series of  $H$ .

The *infinite paths* in  $\mathcal{G}(p, r)$  correspond to the infinite pro- $p$ -groups of coclass  $r$ . Every infinite path in  $\mathcal{G}(p, r)$  defines an infinite pro- $p$ -group of coclass  $r$  via the inverse limit of the groups on the path. Conversely, every infinite pro- $p$ -group of coclass  $r$  yields an infinite path in  $\mathcal{G}(p, r)$  via its lower central series quotients of coclass  $r$ .

The *coclass tree*  $\mathcal{T}(S)$  in  $\mathcal{G}(p, r)$  associated with an infinite pro- $p$ -group  $S$  of coclass  $r$  is the subtree of  $\mathcal{G}(p, r)$  which contains all descendants of the quotient  $S_{/u}$  where  $u$  is the primary root of  $S$ . The groups  $S_{/u}, S_{/u+1}, \dots$  define an infinite path through  $\mathcal{T}(S)$ . The definition of primary root implies that this is the only infinite path in  $\mathcal{T}(S)$  starting at the root of this tree.

The *depth* of a group  $G$  in a coclass tree  $\mathcal{T}(S)$  is its distance to the infinite path in  $\mathcal{T}(S)$ . The *shaved tree*  $\mathcal{T}_k(S)$  is the subtree of  $\mathcal{T}(S)$  containing all groups of depth at most  $k$ . The *depth* of  $\mathcal{T}(S)$  (or  $\mathcal{T}_k(S)$ ) is the maximal depth of the groups in the tree. The depth of  $\mathcal{T}_k(S)$  is bounded by  $k$ , while the depth of  $\mathcal{T}(S)$  can be finite or infinite. Note that the depth of every coclass tree in  $\mathcal{G}(2, r)$  and in  $\mathcal{G}(3, 1)$  is known to be finite, while every other graph  $\mathcal{G}(p, r)$  contains coclass trees of unbounded depth.

As there are only finitely many isomorphism types of infinite pro- $p$ -groups of coclass  $r$ , it follows that  $\mathcal{G}(p, r)$  contains only finitely many different coclass trees. Further, it is not difficult to show that there are only finite many groups in  $\mathcal{G}(p, r)$  which are not contained in any coclass tree.

### 3 Graph periodicity

The periodicity of the shaved coclass trees is proved in [3] and [7]. It asserts that for each integer  $k$  and each infinite pro- $p$ -group  $S$  of finite coclass the shaved coclass tree  $\mathcal{T}_k(S)$  is virtually periodic. For a more precise description, let  $d$  be the dimension of  $S$  and let  $\mathcal{T}_{k,j}(S)$  be the subtree of all descendants of  $S_{/j}$  in  $\mathcal{T}_k(S)$ . Then there exists a  $j \in \mathbb{N}$  such that there exists a graph isomorphism

$$\pi : \mathcal{T}_{k,j}(S) \mapsto \mathcal{T}_{k,j+d}(S).$$

This graph isomorphism allows to define infinite sequences of finite  $p$ -groups of coclass  $r$  in  $\mathcal{T}_k(S)$ . Consider all the (finitely many) groups  $G$  in  $\mathcal{T}_{k,j}(S) \setminus \mathcal{T}_{k,j+d}(S)$ . For each such group  $G$ , define an infinite sequence  $(G_0, G_1, \dots)$  via taking  $G_0 = G$  and  $G_{i+1}$  as the image of  $G_i$  under the graph isomorphism  $\pi$ .

This result asserts that  $\mathcal{T}_k(S)$  consists of finitely many infinite sequences (corresponding one-to-one to the groups in  $\mathcal{T}_{k,j}(S) \setminus \mathcal{T}_{k,j+d}(S)$ ) and the finitely many other groups (contained in  $\mathcal{T}(S) \setminus \mathcal{T}_k(S)$ ). However, so far the result is practically useless in group theoretic applications, as it is based on a purely graph theoretic isomorphism.

In the remainder of this paper we outline an alternative group theoretic definition for the infinite sequences in  $\mathcal{T}_k(S)$ .

### 4 A first overview

Suppose we are given an infinite pro- $p$ -group  $S$  with coclass  $r$ , dimension  $d$  and primary root  $u$  and an integer  $k$ . This section provides a first overview on the construction of infinite sequences in the shaved coclass tree  $\mathcal{T}_k(S)$ . Following the ideas of [7], we proceed in two steps.

First, we choose a set  $\mathcal{P} = \mathcal{P}(S, k)$  consisting of certain tuples of integers of the form  $(l, e)$  with  $l = l(S, k)$  and  $e = e(S, l)$ . A necessary condition for all tuples is that  $l \geq u$  holds. The entries  $l$  are called *secondary roots* and the entries  $e$  are called *offsets*.

Then, we consider each pair  $(l, e)$  in  $\mathcal{P}$  in turn and determine all infinite sequences in  $\mathcal{T}_k(S)$  associated to it. For this purpose, let  $A_i = A_i(l, e) := \gamma_l(S) / \gamma_{l+e+id}(S)$  and note that  $A_i$  is finite abelian of order  $p^{e+id}$ , as  $l \geq u$ . The natural conjugation action of  $S$  on  $A_i$  allows to consider  $A_i$  as an  $S_{/l}$ -module. We now construct infinite sequences of groups  $(G_i \mid i \in \mathbb{N}_0)$  using extension theory: the group  $G_i$  is an extension of  $A_i$  by  $S_{/l}$  via a certain special cocycle  $\delta_i$ .

The following questions remain to be discussed.

#### Questions:

- (1) *How do we choose a suitable set  $\mathcal{P}$ ?*
- (2) *How do we choose the cocycles  $\delta_i$  for the extensions?*
- (3) *Why does that yield a complete set of infinite sequences and is that set irredundant?*

Before we discuss these questions in the subsequent sections of this paper, we exhibit some elementary comments on the construction.

**1 Remark:** *The orders of the groups in an infinite sequence  $(G_i \mid i \in \mathbb{N}_0)$  associated with the infinite pro- $p$ -group  $S$  and the pair  $(l, e)$  can be read off as*

$$|G_0| = |S/l|p^e = p^{r+1+l+e} \quad \text{and} \quad |G_{i+1}| = p^d |G_i| \quad \text{for } i \geq 0.$$

Further, it is not difficult to exhibit not only the order, but also the isomorphism types of the groups  $A_i$ . The following lemma shows that these are always nearly homocyclic and they are homocyclic if and only if the offset  $e$  is divisible by the dimension  $d$ .

**2 Lemma:** *Let  $S$  be an infinite pro- $p$ -group of coclass  $r$ , dimension  $d$  and primary root  $u$ . Let  $l \geq u$  and write  $e = qd + t$  for some  $q, t \in \mathbb{N}_0$  with  $0 \leq t < d$ . Then the group  $A_i = \gamma_l(S)/\gamma_{l+e+id}(S)$  has the abelian invariants  $(a_1, \dots, a_t, a_{t+1}, \dots, a_d)$  with  $a_j = p^{i+q+1}$  for  $1 \leq j \leq t$  and  $a_j = p^{i+q}$  for  $t < j \leq d$ .*

*Proof:* The definition of primary root implies that  $[\gamma_j(S) : \gamma_{j+1}(S)] = p$  for all  $j \geq l$ . This yields that the groups  $\gamma_j(S)$  are the only  $S$ -normal subgroups in  $\gamma_l(S)$ . As  $\gamma_l(S) \cong \mathbb{Z}_p^d$ , it follows that  $\gamma_l(S)^{p^j} = \gamma_{l+jd}(S)$ . This yields the desired result.  $\bullet$

The choice of the cocycles will induce that all groups in an infinite sequence  $(G_i \mid i \in \mathbb{N}_0)$  have the same depth in  $\mathcal{T}_k(S)$ . The following picture sketches their layout.

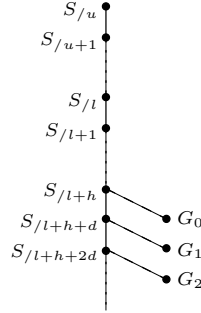


Figure 1: Sketch of infinite sequence in  $\mathcal{T}_k(S)$  with depth  $e - h \leq k$ .

## 5 Choosing a set $\mathcal{P}(S, k)$

In this section we assume that we are given an infinite pro- $p$ -group  $S$  of coclass  $r$  and an integer  $k$ . Let  $d$  be the dimension of  $S$  and let  $u$  be its primary root. Our aim is to determine a set  $\mathcal{P}(S, k)$  such that every infinite sequence in  $\mathcal{T}_k(S)$  can be constructed from exactly one entry in this set.

**3 Definition:** *An integer  $l$  is a secondary root for  $S$  and  $k$  if  $l \geq u$  holds and, secondly, if every descendant  $G$  of  $S/l$  in  $\mathcal{T}_k(S)$  is isomorphic to an extension of  $\gamma_l(S)/\gamma_{d(G)+1}(S)$  by  $S/l$ .*

If a group  $G$  is a descendant of  $S_{/l}$  in  $\mathcal{T}_k(S)$ , then  $G_{/l} \cong S_{/l}$  follows. This allows to consider  $\gamma_l(G)$  as module for  $S_{/l}$  via the natural conjugation action of  $G_{/l}$ . The second condition in Definition 5 is thus equivalent to the following: every group  $G$  in  $\mathcal{T}_{k,l}(S)$  satisfies that  $\gamma_l(G)$  is isomorphic as  $S_{/l}$ -module to  $\gamma_l(S)/\gamma_{cl(G)+1}(S)$ .

It is not obvious that a secondary root always exists. This is a deep result of coclass theory and we refer to Chapters 5 and 11 of [9] for background. We note that the results there also imply the following remark.

**4 Remark:** *Let  $S$  be an infinite pro- $p$ -group of finite coclass and let  $k \in \mathbb{N}$ . If  $l$  is a secondary root for  $S$  and  $k$ , then every  $g \geq l$  is also one.*

While a primary root  $u$  is usually not difficult to determine, it is often not straightforward to find an integer which satisfies the second condition, as this requires detailed knowledge about almost all groups in the graph  $\mathcal{T}_k(S)$ . This problem is addressed in the following theorem which follows directly from Theorem 15 in [7].

**5 Theorem:** *Let  $S$  be an infinite pro- $p$ -group of coclass  $r$  and dimension  $d$ . Let  $u$  be the primary root of  $S$  and let  $k \in \mathbb{N}$ . Then every integer  $l$  with  $l \geq u - 1 + \max\{p^r, \frac{3}{2}(k + d)\}$  is a secondary root for  $S$  and  $k$ .*

Suppose now that a secondary root  $l$  for  $S$  and  $k$  is given. To shorten notation let  $R = S_{/l}$  and  $T = \gamma_l(S)$ . Let  $H$  be the kernel of the action of  $R$  on  $T$  with commutator quotient  $H/H'$  and Schur multiplier  $M(H)$ . Define the integers  $a$  and  $b$  by  $p^a = \exp(H/H')$  and  $p^b = \exp(M(H))$ .

**6 Definition:** *An integer  $e$  is an offset for  $S$  and  $l$  if*

$$e \geq \max\{2d(a + b + 1), d(l + r - 1)\}.$$

The following theorem simplifies the determination of offsets. It follows directly from Theorem 29 in [7].

**7 Theorem:** *Let  $S$  be an infinite pro- $p$ -group of coclass  $r$  and let  $l$  be a secondary root for  $S$ . Then every  $e \geq 2d(2l + 2r - 1)$  is an offset for  $S$  and  $l$ .*

It remains to discuss how a complete and irredundant set of secondary roots and associated offsets can be obtained.

**8 Definition:** *We choose  $\mathcal{P}(S, k) = \{(l_1, e_1), \dots, (l_d, e_d)\}$  such that for  $1 \leq i \leq d$  the following conditions hold:*

- $l_i$  is a secondary root for  $S$  and  $k$ ,
- $e_i$  is an offset for  $S$  and  $l_i$ , and
- $l_i + e_i \equiv i \pmod{d}$  for  $1 \leq i \leq d$ .

This provides a set of secondary roots and associated offsets. Its completeness is proved in Section 7 below. Its irredundancy can be observed easily, as the orders of the groups in an infinite sequence associated with  $(l, e)$  depend on the values of  $l + e \pmod{d}$  by Remark 1. Hence every infinite sequence can be associated with at most one pair  $(l, e)$  in  $\mathcal{P}(S, k)$ .

## 6 Constructing infinite sequences

In this section we assume that we have given an infinite pro- $p$ -group  $S$ , an integer  $k$ , a secondary root  $l$  and an offset  $e$ . We show how to construct all infinite sequences in  $\mathcal{T}_k(S)$  associated with  $(l, e)$ .

Let  $T = \gamma_l(S)$  and  $R = S/T$ . The choice for  $l$  implies that  $R$  is a finite  $p$ -group of coclass  $r$  and  $T$  is isomorphic to  $\mathbb{Z}_p^d$ . Denote  $T_i = \gamma_{l+i}(S)$  and  $A_i = T/T_{e+id}$  for  $i \in \mathbb{N}$ . The conjugation action of  $S$  on  $T$  induces that  $A_i$  is an  $R$ -module.

The groups in an infinite sequence  $\mathcal{G} = (G_i \mid i \in \mathbb{N}_0)$  are defined as extensions of  $A_i$  by  $R$  via certain cocycles  $\delta_i \in H^2(R, A_i)$ . We describe a construction for these cocycles below. We consider the groups on the infinite path of  $\mathcal{T}_k(S)$  separately as a first step, since these are a particularly easy case.

### 6.1 Sequences on the infinite path

The projection  $T \rightarrow A_i$  induces a natural map  $\nu_i : H^2(R, T) \rightarrow H^2(R, A_i)$ . Let  $\alpha \in H^2(R, T)$  be an arbitrary, but fixed element which defines the infinite pro- $p$ -group  $S$  as extension of  $T$  by  $R$  and let  $\alpha_i$  denote its image under the projection map. Then  $\alpha_i$  defines  $S_{/l+e+id}$  as extension of  $A_i$  by  $R$ .

**9 Definition:** *The infinite sequence defined by 0 with respect to  $S$  and  $(l, e)$  is the sequence of groups  $(G_i \mid i \in \mathbb{N}_0)$  with  $G_i = S_{/l+e+id}$ .*

Note that this is an infinite sequence of groups on the infinite path of  $\mathcal{T}_k(S)$ . There are  $d$  different infinite sequences of groups on the infinite path. These arise with the construction described here by varying the values for  $(l, e)$ .

### 6.2 Arbitrary infinite sequences

We now consider the construction of arbitrary infinite sequences with secondary root  $l$  and offset  $e$ . This mainly requires the definition of suitable cocycles in  $H^2(R, A_i)$ . We first investigate the structure of this cohomology group. As above, let  $H$  be the kernel of the action of  $R$  on  $T$  and let  $p^a = \exp(H/H')$ . For  $i \in \mathbb{N}_0$  define  $B_i = T_{d(a+1)+id}/T_{e+id}$  and note that  $B_i$  is an  $R$ -invariant subgroup of  $A_i$ . Further, note that Lemma 2 implies that  $A_0 \cong A_i^{p^i}$  so that we can identify  $A_0$  with an  $R$ -invariant subgroup of  $A_i$ . We define

$$\begin{aligned} \pi_i & : H^2(R, A_0) \rightarrow H^2(R, A_i) \text{ induced by inclusion } A_0 \cong A_i^{p^i} \rightarrow A_i, \\ \mu_i & : H^2(R, B_i) \rightarrow H^2(R, A_i) \text{ induced by inclusion } B_i \rightarrow A_i, \\ \nu_i & : H^2(R, T) \rightarrow H^2(R, A_i) \text{ induced by projection } T \rightarrow A_i. \end{aligned}$$

Denote with  $N_i$  and  $M_i$  the images of  $\nu_i$  and  $\mu_i$ , respectively.

**10 Theorem:** *Let  $i \in \mathbb{N}_0$ .*

- a)  $H^2(R, A_i) = N_i \oplus M_i$ .
- b)  $N_i \cong H^2(R, T)$  and  $M_i \cong H^3(R, T_e)$ .
- c)  $\pi_i$  restricts to an isomorphism from  $M_0$  onto  $M_i$ .

*Proof:* Theorem 18 of [7] and the definition of offsets yields that  $H^2(R, A_i) \cong H^2(R, T) \oplus H^3(R, T_{e+id})$ . Powering by  $p^i$  induces an isomorphism  $T_e \cong T_{e+id}$  and thus an isomorphism

$H^3(R, T_e) \cong H^3(R, T_{e+id})$ . The proof of Theorem 18 in [7] implies that the direct component  $H^2(R, T)$  corresponds to the subgroup  $N_i$  of  $H^2(R, A_i)$ . Theorem 19 in [7] shows that the direct component  $H^3(R, T_{e+id})$  corresponds to the subgroup  $M_i$  of  $H^2(R, A_i)$  and that  $\pi_i$  is an isomorphism from  $M_0$  onto  $M_i$ . •

Recall that  $\alpha \in H^2(R, T)$  defines  $S$  as extension of  $T$  by  $R$  and that  $\alpha_i$  is the image of  $\alpha$  under  $\nu_i$ . We often identify  $H^3(R, T_e)$  with  $M_0$  in the following using the isomorphism obtained in Theorem 10 b).

**11 Definition:** *The infinite sequence defined by  $\beta \in H^3(R, T_e) \cong M_0$  with respect to  $S$  and  $(l, e)$  is the sequence of groups  $(G_i \mid i \in \mathbb{N}_0)$  where  $G_i$  is the extension of  $A_i$  by  $R$  via the cocycle class  $\delta_i = \alpha_i + \pi_i(\beta)$ .*

As  $R$  is a finite group, it follows that  $M_0 \cong H^3(R, T)$  is a finite group and thus this construction yields finitely many infinite sequences. We note that different cocycle classes  $\beta$  and  $\beta'$  can yield infinite sequences whose groups are pairwise isomorphic. We will not investigate this problem further and instead refer to [7] for a full solution of the isomorphism problem. We conclude this section with the following remark.

**12 Remark:**  $\mathcal{G}_\beta$  is the infinite sequences on the infinite path if and only if  $\beta = 0$ .

## 7 Completeness and redundancy

In the first part of this section, we show that the construction introduced above yields a complete set of infinite sequences in a shaved coclass tree  $\mathcal{T}_k(S)$ .

**13 Theorem:** *Let  $S$  be an infinite pro- $p$ -group of coclass  $r$  and  $k$  an integer. Let  $\mathcal{P}(S, k)$  be a set of pairs satisfying the conditions of Definition 8. Then almost all groups in  $\mathcal{T}_k(S)$  are contained in an infinite sequence.*

*Proof:* Our proof relies heavily on the results of [7]. Let  $l_0 = \max\{l \mid (l, e) \in \mathcal{P}\}$  and  $e_0 = \max\{e \mid (l, e) \in \mathcal{P}\}$ . Let  $G$  be a group in  $\mathcal{T}_k(S)$  which is a descendant of  $S_{/l_0+e_0}$  and has order at least  $p^{l_0+2e_0+r-1}$ . Note that almost all groups in  $\mathcal{T}_k(S)$  have this form. We show that  $G$  is contained in an infinite sequence.

Let  $|G| = p^{r-1+x}$  for some  $x \geq l_0 + 2e_0$ . Let  $(l, e) \in \mathcal{P}$  with  $l + e \equiv x \pmod{d}$ . Then  $G$  is a descendant of  $S_{/l}$  as well, as  $l \leq l_0$ . Let  $R = S_{/l}$  and write  $T = \gamma_l(S)$  and  $T_i = \gamma_{l+i}(S)$ . Since  $l$  is a secondary root, it follows from Theorem 15 in [7], that  $G$  is an extension of  $T/T_n$  by  $R$  for  $n = x - l$ . Hence there exists a cocycle class  $\gamma$  in  $H^2(R, T/T_n)$  defining  $G$ . Note that  $x \geq l_0 + e_0 \geq l + e$  and  $x \equiv l + e \pmod{d}$ . Thus there exists an  $i$  with  $x = l + e + id$  and hence  $n = e + id$ . Since  $l$  is a secondary root and  $e$  is an offset for it, Theorem 10 applies and  $H^2(R, T/T_n) \cong H^2(R, T) \oplus H^3(R, T_e)$ . Let  $(\delta, \beta)$  be the image of  $\gamma$  under this isomorphism.

Note that  $n = x - l \geq l_0 - l + 2e_0 \geq 2e_0$ . Choosing  $i = l_0 + e_0$  we obtain that  $n$  and  $i$  satisfy the conditions of Theorem 26 in [7]. As  $G$  is a descendant of  $S_{/i}$ , it follows from Theorem 26b) in [7] that  $\delta$  defines  $S$  as extension of  $T$  by  $R$ . Further, Theorem 26a) of [7] asserts that  $\delta$  is equivalent to  $\alpha$ ; that is, there exists an automorphism which maps  $\delta$  to  $\alpha$  without changing the isomorphism type of  $G$ . Hence w.l.o.g. we can replace  $\delta$  by  $\alpha$ . This yields that  $G$  is a group in the infinite sequence defined by  $(l, e)$  and the cocycle class  $\beta \in H^3(R, T_e) \cong M_0$ . •



Our construction of infinite sequences is at least partially redundant, as we summarize in the following theorem.

**14 Theorem:** *Let  $S$  be an infinite pro- $p$ -group of coclass  $r$  and  $k$  an integer.*

- a) *Every infinite sequence in  $\mathcal{T}_k(S)$  is associated with a unique pair  $(l, e) \in \mathcal{P}$  and a (not necessarily unique) element  $\beta \in H^3(S/l, \gamma_{l+e}(S))$ .*
- b) *Let  $\beta$  and  $\beta'$  be two elements of  $H^3(S/l, \gamma_{l+e}(S))$  defining the infinite sequences  $(G_i(\beta) \mid i \in \mathbb{N}_0)$  and  $(G_i(\beta') \mid i \in \mathbb{N}_0)$ . Then  $G_i(\beta) \cong G_i(\beta')$  holds for all  $i \in \mathbb{N}_0$  if and only if  $G_0(\beta) \cong G_0(\beta')$  holds.*

*Proof:* a) The uniqueness of the pair  $(l, e)$  follows from the definition of  $\mathcal{P}$  and the fact that the orders of the groups in an infinite sequence depend on  $(l, e)$  modulo  $d$ .

b) This follows from Theorem 25 in [7] by using the definition of offset. •

Our construction of infinite sequences underpins the periodicity graph isomorphism as described in Section 3. More precisely, if  $S$ ,  $k$  and  $\mathcal{P}(S, k)$  are given, then with  $l_0 = \max\{l \mid (l, e) \in \mathcal{P}\}$  and  $e_0 = \max\{e \mid (l, e) \in \mathcal{P}\}$  and  $j = l_0 + 2e_0$  a graph isomorphism is induced via the infinite sequences.

## 8 Parametrised presentations

It is proved in [7] that the groups in an infinite sequence can be defined by a single parametrised presentation. We exhibit here that these presentations have a particularly nice form if the offset is divisible by the dimension of the underlying pro- $p$ -group.

Let  $S$  be an infinite pro- $p$ -group of dimension  $d$  and let  $(l, e)$  be a pair of secondary root and offset. Let  $R = S/l = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$  and let  $t_1, \dots, t_d$  be a (topological) generating set for  $T = \gamma_l(S)$ . To shorten notation, we write  $t^{\bar{v}}$  for  $t_1^{v_1} \dots t_d^{v_d}$  where  $\bar{v} = (v_1, \dots, v_d) \in \mathbb{Z}_p^d$ . We express the action of  $R$  on  $T$  by vectors  $\bar{v}_{hj} \in \mathbb{Z}_p^d$  with  $t_j^{g_h} = t^{\bar{v}_{hj}}$  for  $1 \leq h \leq n$  and  $1 \leq j \leq d$ .

The pro- $p$ -group  $S$  is an extension of  $T$  by  $R$  and thus it has a pro- $p$ -presentation which exhibits this extension structure. This has the generators  $g_1, \dots, g_n, t_1, \dots, t_d$  and its relations have the following form for some vectors  $\bar{v}_1, \dots, \bar{v}_m \in \mathbb{Z}_p^d$ :

$$\begin{aligned} r_j &= t^{\bar{v}_j} \text{ for } 1 \leq j \leq m, \\ t_j^{g_h} &= t^{\bar{v}_{hj}} \text{ for } 1 \leq j \leq d, 1 \leq h \leq n, \\ [t_j, t_h] &= 1 \text{ for } 1 \leq j, h \leq d. \end{aligned}$$

Recall that we assume that  $e$  is divisible by  $d$ . Then the pro- $p$ -presentation for  $S$  can be modified readily to a presentation for the finite  $p$ -group  $S_{/l+e+id}$  by adding the relations  $t_j^{p^{e/d+i}}$  for  $1 \leq j \leq d$ . The following theorem shows how the presentation can be modified to a presentation for a group in an infinite sequence.

**15 Theorem:** *Let  $(G_i \mid i \in \mathbb{N}_0)$  be an infinite sequence associated with  $S$  and  $(l, e)$ . Then there exist vectors  $\bar{w}_1, \dots, \bar{w}_m \in \mathbb{Z}_p^d$  so that every group  $G_i$  can be defined by a presentation on the generators  $g_1, \dots, g_n, t_1, \dots, t_d$  with relations of the form*

$$\begin{aligned} r_j &= t^{\bar{v}_j} t^{p^i \bar{w}_j} \text{ for } 1 \leq j \leq m, \\ t_j^{g_h} &= t^{\bar{v}_{hj}} \text{ for } 1 \leq j \leq d, 1 \leq h \leq n, \\ [t_j, t_h] &= 1 \text{ for } 1 \leq j, h \leq d, \\ t_j^{p^{e/d+i}} &= 1 \text{ for } 1 \leq j \leq d. \end{aligned}$$



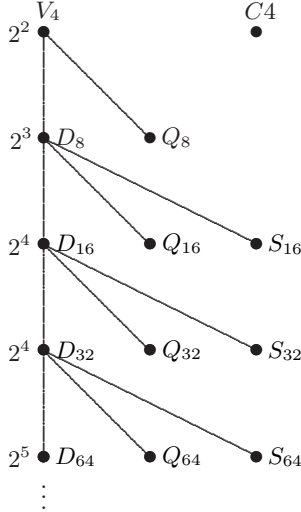
*Proof:* This follows directly from the construction on the infinite sequences: these are extensions of  $A_i = T/\gamma_{l+e+id}(S) = T/T^{p^{t+i}}$  by  $R$  and the considered cocycle classes  $\delta_i$  have the form  $\alpha_i + \pi_i(\beta)$ , where  $\pi_i$  is obtained by powering with  $p^i$ . The vectors  $\bar{v}_1, \dots, \bar{v}_m$  correspond to the cocycle class  $\alpha_i$  and the vectors  $\bar{w}_1, \dots, \bar{w}_m$  correspond to  $\pi_i(\beta)$ . •

We note that the exponents in the presentation for the groups  $G_i$  are defined as elements in  $\mathbb{Z}_p$ , but the relations  $t_j^{p^{e/d+i}}$  allow us to consider them as elements in  $\mathbb{Z}_p/p^{e/d+i}\mathbb{Z}_p$ .

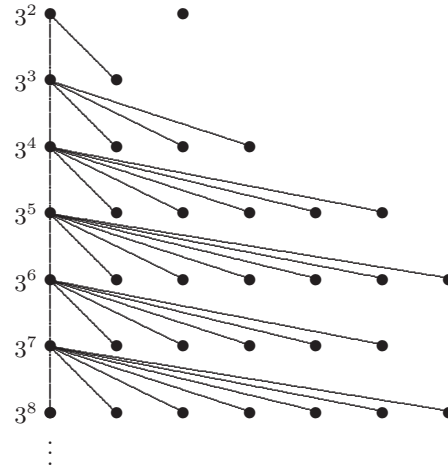
**16 Remark:** Every infinite sequence associated with  $S$  and  $(l, e)$  can thus be defined by a presentation of  $S$  as extension of  $T$  by  $R$  and a list of vectors  $\bar{w}_1, \dots, \bar{w}_m \in \mathbb{Z}_p^d$ .

## 9 The graphs $\mathcal{G}(2, 1)$ and $\mathcal{G}(3, 1)$

The graphs  $\mathcal{G}(2, 1)$  and  $\mathcal{G}(3, 1)$  have been known for a long while. The groups in these graphs have been classified by Blackburn [1]. We exhibit the graphs in the following picture.



The graph  $\mathcal{G}(2, 1)$ .



The graph  $\mathcal{G}(3, 1)$ .

We demonstrate a general method to explicitly compute the pairs  $\mathcal{P}(S, k)$  as in Section 5 and explicitly determine the cohomology groups as in Section 6 in the example  $\mathcal{G}(2, 1)$ . We then show how the infinite sequences can be described in a highly compact form using the presentations of Section 8 in the example  $\mathcal{G}(3, 1)$ .

Both graphs  $\mathcal{G}(2, 1)$  and  $\mathcal{G}(3, 1)$  contain a single coclass tree and this has depth 1. Hence we use  $k = 1$  in all example computations of this section.

### 9.1 The graph $\mathcal{G}(2, 1)$

Let  $S$  be the infinite pro-2-group of coclass 1. Then  $S = \mathbb{Z}_2 \rtimes C_2$ , where the cyclic group  $C_2$  acts by additive inversion on the 2-adic integers  $\mathbb{Z}_2$ . Thus  $S$  has dimension  $d = 1$ . Further, its primary root is  $u = 2$ , as  $\gamma_2(S) = 2\mathbb{Z}_2 \cong \mathbb{Z}_2$  is torsion-free abelian and  $S_{/2} \cong C_2 \times C_2$  has coclass 1.

**Secondary root.** Every  $l \geq 2$  is a secondary root for  $S$  and  $k$ . This can be observed by an explicit investigation of all groups in the graph using the classification by Blackburn [1]. Note that Theorem 5 yields that every  $l \geq 4$  is a possible secondary root.

**Offset.** We determine the offsets for  $l = 2$ . Let  $R = S_{/2}$  and  $T = \gamma_2(S)$ . The kernel  $H$  of the action of  $R$  on  $T$  is cyclic of order 2. Thus we obtain  $a = 1$  and  $b = 0$  for the logarithmic exponents of the commutator quotient and Schur multiplier of  $H$ . It follows that every  $e \geq 4$  is an offset for  $l = 2$ . Theorem 7 yields that every  $e \geq 10$  is an offset for  $l = 2$ .

**Pairs.** We choose  $\mathcal{P} = \{(2, 4)\}$  and note that this satisfies the conditions of Definition 8. Thus we consider  $l = 2$  and  $e = 4$  and, accordingly,  $R = S_{/2}$  and  $T \cong \mathbb{Z}_2$  in the following. This yields that  $A_i = T/T_{e+id} = T/T_{4+i}$  is cyclic of order  $2^{4+i}$ .

**Cohomology.** We determine the cohomology group  $H^2(R, A_i)$  explicitly with the method of Section 8.7.2 of [8]. For this purpose we use that every element  $\epsilon$  in  $Z^2(R, A_i)$  defines an extension of  $A_i$  by  $R$ . This extension has a presentation on the generators  $g_1, g_2, t$  with relations

$$g_1^2 = t^x, g_2^2 = t^y, g_2^{g_1} = g_2 t^z, t^{g_1} = t^{-1}, t^{g_2} = t, t^{2^{4+i}} = 1.$$

for certain  $x, y, z \in \{0, \dots, 2^{4+i} - 1\}$ . Thus we obtain a map

$$Z^2(R, A_i) \rightarrow (\mathbb{Z}_2/2^{4+i}\mathbb{Z}_2)^3 : \epsilon \mapsto (x, y, z).$$

Let  $\hat{Z}_i$  and  $\hat{B}_i$  denote the images of  $Z^2(R, A_i)$  and  $B^2(R, A_i)$ , respectively, under this map. An explicit calculation (see Section 8.7.2 of [8] for details) yields that

$$\hat{Z}_i = \langle (2^{3+i}, 0, 0), (0, 0, 2^{3+i}), (0, 1, -1) \rangle \quad \text{and} \quad \hat{B}_i = 2\hat{Z}_i.$$

By Lemma 8.47 in [8], we obtain that  $H^2(R, A_i) \cong \hat{Z}_i/\hat{B}_i$  and thus  $H^2(R, A_i)$  is elementary abelian of order  $2^3$  for every  $i \in \mathbb{N}_0$ .

**The image of  $H^2(R, T)$ .** A presentation for  $S$  as extension of  $T$  by  $R$  is exhibited by the following presentation on the generators  $g_1, g_2, t$  with relations

$$g_1^2 = 1, g_2^2 = t, g_2^{g_1} = g_2 t^{-1}, t^{g_1} = t^{-1}, t^{g_2} = t.$$

Comparing this presentation with the presentation for the extensions of  $A_i$  by  $R$  above shows that  $\hat{\alpha}_i = (0, 1, -1)$  defines the group  $S_{/6+i}$  as extension of  $A_i$  by  $R$ .

**The image of  $H^2(R, B_i)$ .** We find that  $B_i = T_3/T_{4+i} \leq A_i$  and thus  $B_i = 2^3 A_i$ . Thus the image  $\hat{M}_i$  of  $H^2(R, B_i)$  in our construction for  $H^2(R, A_i)$  can be read off easily as

$$\hat{M}_i = 2^{3+i} \langle (1, 0, 0), (0, 0, 1) \rangle + \hat{B}_i/\hat{B}_i.$$

Hence  $\hat{M}_i$  is elementary abelian of order 4 for every  $i \in \mathbb{N}_0$ . Recall that  $M_i$  is obtained from  $M_0$  by powering with  $p^i$ . As we use additive notation, it follows that  $\hat{M}_i$  is obtained from  $\hat{M}_0$  by multiplication with  $2^i$ .

**The infinite sequences.** There are  $4 = |M_0|$  infinite sequences arising from our construction in this case. These are exhibited in the following table which lists for each sequence its defining element  $\hat{\beta} \in \hat{M}_0 \cong H^3(R, T_e)$ , the elements  $\hat{\delta}_i = \hat{\alpha}_i + \hat{\beta}_i$  defining the groups  $G_i(\beta)$  and the names of the obtained groups. The latter shows that there are two different cocycles which yield isomorphic sequences.

$\hat{\beta}$	$\hat{\delta}_i$	name
$(0, 0, 0)$	$(0, 1 - 1)$	dihedral groups
$(2^3, 0, 0)$	$(0, 1 - 1) + 2^i(2^3, 0, 0)$	quaternion groups
$(2^3, 0, 2^3)$	$(0, 1 - 1) + 2^i(2^3, 0, 2^3)$	semidihedral groups
$(0, 0, 2^3)$	$(0, 1 - 1) + 2^i(0, 0, 2^3)$	semidihedral groups

## 9.2 The graph $\mathcal{G}(3, 1)$

Let  $S$  be the infinite pro-3-group of coclass 1. Then  $S = \mathbb{Z}_3^2 \rtimes C_3$  and hence  $S$  has dimension  $d = 2$ . The classification by Blackburn shows that  $u = 2$  is the primary root for  $S$  and  $l = 3$  is the minimal secondary root. We choose

$$\mathcal{P} = \{(3, 12), (4, 14)\}$$

as set of pairs. This choice yields that  $d \mid e$  for both offsets  $e$  and hence Theorem 15 can be used to describe the infinite sequences corresponding to these pairs. This is done in the following.

**The case  $(l, e) = (3, 12)$ .** A presentation of  $S$  as extension of  $\gamma_3(S)$  by  $S/3$  has the generators  $g_1, g_2, g_3, t_1, t_2$  and the relations

$$\begin{aligned} g_1^3 &= 1, g_2^3 = t_1 t_2, g_2^{g_1} = g_2 g_3, g_3^3 = t_1^{-3} t_2^{-2}, g_3^{g_1} = g_3 t_1^2 t_2, g_3^{g_2} = g_3, \\ t_1^{g_1} &= t_1 t_2, t_1^{g_2} = t_1, t_1^{g_3} = t_1, t_2^{g_1} = t_1^{-3} t_2^{-2}, t_2^{g_2} = t_2, t_2^{g_3} = t_2, t_2^{t_1} = t_2. \end{aligned}$$

The first 3 generators and the first 6 relations in this presentation of  $S$  correspond to a presentation of  $R$ . By Remark 16, each infinite sequence associated with this case can thus be described by a list of vectors  $\mathcal{W} = (\overline{w}_1, \dots, \overline{w}_6)$  with  $\overline{w}_i \in \mathbb{Z}_3^2$ . Let

$$\begin{aligned} \mathcal{W}_1 &= 3^5((0, 1), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)), \\ \mathcal{W}_2 &= 3^5((0, 0), (0, 0), (0, 0), (0, 0), (0, 1), (0, 0)), \\ \mathcal{W}_3 &= 3^5((0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 1)). \end{aligned}$$

Then the lists of vectors  $\mathcal{W}$  defining the infinite sequences are exactly the  $\mathbb{Z}_3$ -linear combinations of  $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ . As  $e/d = 6$ , we can work modulo  $3^6 \mathbb{Z}_3$ . Thus there are 27 linear combinations of lists of vectors. Among these, the elements in  $\{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_2 + \mathcal{W}_3, \mathcal{W}_1 + 2\mathcal{W}_3\}$  yield the six non-isomorphic infinite sequences.

**The case  $(l, e) = (4, 14)$ .** A presentation of  $S$  as extension of  $\gamma_4(S)$  by  $S/4$  has the generators  $g_1, g_2, g_3, g_4, t_1, t_2$  and the relations

$$\begin{aligned} g_1^3 &= 1, g_2^3 = g_4^2, g_2^{g_1} = g_2 g_3, g_3^3 = t_1^2 t_2^2, g_3^{g_1} = g_3 g_4 t_2, g_3^{g_2} = g_3, g_4^3 = t_1^{-2} t_2^{-3}, g_4^{g_1} = \\ g_4 t_1 t_2, g_4^{g_2} &= g_4, g_4^{g_3} = g_4, t_1^{g_1} = t_1 t_2^3, t_1^{g_2} = t_1, t_1^{g_3} = t_1, t_1^{g_4} = t_1, t_2^{g_1} = t_1^{-1} t_2^{-2}, t_2^{g_2} = \\ t_2, t_2^{g_3} &= t_2, t_2^{g_4} = t_2, t_2^{t_1} = t_2, \end{aligned}$$

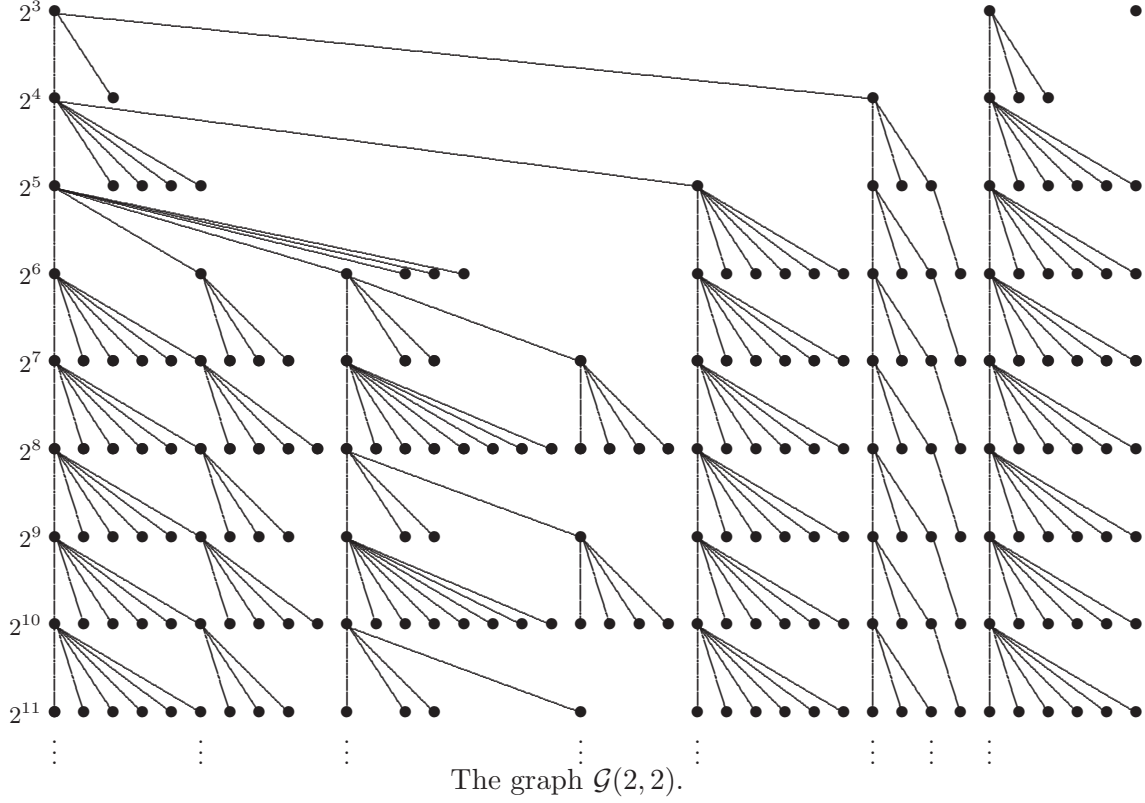
Let  $\overline{\mathcal{W}} = \langle \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \rangle_{\mathbb{Z}_3}$  with

$$\begin{aligned} \mathcal{W}_1 &= 3^6((1, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)), \\ \mathcal{W}_2 &= 3^6((0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (1, 0), (0, 0), (0, 0), (0, 0), (0, 0)), \\ \mathcal{W}_3 &= 3^6((0, 0), (0, 0), (0, 0), (1, 0), (0, 0), (-1, 0), (0, 0), (2, 0), (0, 0), (0, 0), (0, 0)). \end{aligned}$$

Then every  $\mathcal{W} \in \overline{\mathcal{W}}$  defines an infinite sequence. As  $e/d = 7$ , we can work modulo  $3^7 \mathbb{Z}_3$  which yields 27 elements in  $\overline{\mathcal{W}}$ . The elements in  $\{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_2 + \mathcal{W}_3, \mathcal{W}_1 + 2\mathcal{W}_3, 2\mathcal{W}_2 + 2\mathcal{W}_3\}$  yield the seven different infinite sequences.

## 10 The graph $\mathcal{G}(2, 2)$

The 2-groups of coclass 2 have first been investigated by Newman & O'Brien [11]. We recall the explicit picture of the resulting graph  $\mathcal{G}(2, 2)$  in the following.



There are essentially five different infinite paths in  $\mathcal{G}(2, 2)$  which correspond to the five infinite pro-2-groups of coclass 2. In the following table we list some parameters for each of these five groups.

group	dimension $d$	depth $k$	primary root $u$	minimal secondary root $l$
$S_1$	2	2	5	5
$S_2$	2	2	5	5
$S_3$	1	1	4	4
$S_4$	1	2	3	3
$S_5$	1	1	2	2

We exhibit presentations for each of the infinite sequences of  $\mathcal{G}(2, 2)$  in the same format as for  $\mathcal{G}(3, 1)$ . The following table lists the pairs we consider and gives an overview on the cohomology and the infinite sequences in each case.

group	$(l, e)$	$H^2(R, A_i)$	$ H^3(R, T_e) $	# non-isom sequences
$S_1$	(5, 12)	$C_2^4 \times C_4^2$	16	10
$S_1$	(6, 14)	$C_2^4 \times C_4 \times C_8$	16	9
$S_2$	(5, 12)	$C_2^6 \times C_4$	32	12
$S_2$	(6, 14)	$C_2^3 \times C_8$	8	4
$S_3$	(4, 6)	$C_2^3 \times C_8$	8	6
$S_4$	(3, 8)	$C_2 \times C_4^2$	8	4
$S_5$	(2, 6)	$C_2^6$	16	6

We use Remark 16 to describe parametrised presentations for each infinite sequence. In each case, we outline the relevant presentations for  $S$  and exhibit a set of lists of vectors

$\mathcal{W} = (\overline{w}_1, \dots, \overline{w}_m)$  with  $\overline{w}_i \in \mathbb{Z}_2^d$  defining the infinite sequences. Note that  $\overline{w}_i$  corresponds to the  $i$ th relation of  $S$  in all cases. To shorten notation, we sometimes replace a 0-vector  $\overline{w}_i$  by a single dot and we eliminate  $\overline{w}_j, \dots, \overline{w}_m$  if these vectors are all 0-vectors.

**$S_1$  with (5, 12).** Generators  $g_1, \dots, g_6, t_1, t_2$  and relations

$$\begin{aligned} g_1^2 &= g_4, g_2^2 = 1, g_2^{g_1} = g_2 g_3, g_3^2 = g_6, g_3^{g_1} = g_3 g_5, g_3^{g_2} = g_3 g_6 t_2, g_4^2 = 1, g_4^{g_1} = \\ &g_4, g_4^{g_2} = g_4 g_5 g_6 t_1^{-1}, g_4^{g_3} = g_4 g_6, g_5^2 = t_1 t_2, g_5^{g_1} = g_5 g_6 t_1^{-1}, g_5^{g_2} = g_5 t_1^{-1} t_2^{-1}, g_5^{g_3} = \\ &g_5, g_5^{g_4} = g_5 t_1^{-1} t_2^{-1}, g_6^2 = t_2^{-1}, g_6^{g_1} = g_6 t_1 t_2, g_6^{g_2} = g_6 t_2, g_6^{g_3} = g_6, g_6^{g_4} = g_6 t_2, g_6^{g_5} = \\ &g_6, t_1^{g_1} = t_1 t_2, t_1^{g_2} = t_1^{-1}, t_1^{g_3} = t_1, t_1^{g_4} = t_1^{-1}, t_1^{g_5} = t_1, t_1^{g_6} = t_1, t_2^{g_1} = t_1^{-2} t_2^{-1}, t_2^{g_2} = \\ &t_2^{-1}, t_2^{g_3} = t_2, t_2^{g_4} = t_2^{-1}, t_2^{g_5} = t_2, t_2^{g_6} = t_2, t_2^{t_1} = t_2. \end{aligned}$$

$$\begin{aligned} \mathcal{W}_1 &= 2^5(\cdot, (1, 0), \cdot, \cdot, (0, 0), (0, 1), (0, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 0)), \\ \mathcal{W}_2 &= 2^5(\cdot, (0, 1), \cdot, \cdot, (0, 0), (0, 0), (0, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 0)), \\ \mathcal{W}_3 &= 2^5(\cdot, (0, 0), \cdot, \cdot, (0, 1), (0, 0), (0, 0), \cdot, (0, 1), (0, 1), \cdot, (0, 1)), \\ \mathcal{W}_4 &= 2^5(\cdot, (0, 0), \cdot, \cdot, (0, 0), (0, 0), (0, 1), \cdot, (0, 0), (0, 0), \cdot, (0, 0)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_1 + \mathcal{W}_3, \mathcal{W}_1 + \mathcal{W}_4, \mathcal{W}_2 + \mathcal{W}_3, \mathcal{W}_2 + \mathcal{W}_4, \mathcal{W}_3 + \mathcal{W}_4\} \end{aligned}$$

**$S_1$  with (6, 14).** Generators  $g_1, \dots, g_6, t_1, t_2$  and relations

$$\begin{aligned} g_1^2 &= g_4, g_2^2 = 1, g_2^{g_1} = g_2 g_3, g_3^2 = g_5^2 g_6, g_3^{g_1} = g_3 g_5, g_3^{g_2} = g_3 g_5^2 g_6 t_1^{-1} t_2^{-1}, g_4^2 = \\ &1, g_4^{g_1} = g_4, g_4^{g_2} = g_4 g_5 g_6 t_1^{-1} t_2^{-1}, g_4^{g_3} = g_4 g_5^2 g_6, g_5^4 = t_1^{-1}, g_5^{g_1} = g_5 g_6 t_1^{-1} t_2^{-1}, g_5^{g_2} = \\ &g_5^3 t_1, g_5^{g_3} = g_5, g_5^{g_4} = g_5^3 t_1, g_6^2 = t_1^2 t_2, g_6^{g_1} = g_5^2 g_6 t_2, g_6^{g_2} = g_6 t_1^{-2} t_2^{-1}, g_6^{g_3} = g_6, g_6^{g_4} = \\ &g_6 t_1^{-2} t_2^{-1}, g_6^{g_5} = g_6, t_1^{g_1} = t_1 t_2^2, t_1^{g_2} = t_1^{-1}, t_1^{g_3} = t_1, t_1^{g_4} = t_1^{-1}, t_1^{g_5} = t_1, t_1^{g_6} = t_1, t_2^{g_1} = \\ &t_1^{-1} t_2^{-1}, t_2^{g_2} = t_2^{-1}, t_2^{g_3} = t_2, t_2^{g_4} = t_2^{-1}, t_2^{g_5} = t_2, t_2^{g_6} = t_2, t_2^{t_1} = t_2. \end{aligned}$$

$$\begin{aligned} \mathcal{W}_1 &= 2^6(\cdot, (1, 0), \cdot, \cdot, (0, 0), (0, 0), (0, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 0)), \\ \mathcal{W}_2 &= 2^6(\cdot, (0, 1), \cdot, \cdot, (0, 0), (1, 0), (0, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 0)), \\ \mathcal{W}_3 &= 2^6(\cdot, (0, 0), \cdot, \cdot, (1, 0), (0, 0), (0, 0), \cdot, (1, 0), (1, 0), \cdot, (1, 0)), \\ \mathcal{W}_4 &= 2^6(\cdot, (0, 0), \cdot, \cdot, (0, 0), (0, 0), (1, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 0)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_1 + \mathcal{W}_3, \mathcal{W}_1 + \mathcal{W}_4, \mathcal{W}_2 + \mathcal{W}_4, \mathcal{W}_2 + \mathcal{W}_3\}. \end{aligned}$$

**$S_2$  with (5, 12).** Generators  $g_1, \dots, g_6, t_1, t_2$  and relations

$$\begin{aligned} g_1^2 &= g_4, g_2^2 = 1, g_2^{g_1} = g_2 g_3, g_3^2 = 1, g_3^{g_1} = g_3 g_5, g_3^{g_2} = g_3, g_4^2 = 1, g_4^{g_1} = g_4, g_4^{g_2} = \\ &g_4 g_5 t_1^{-1}, g_4^{g_3} = g_4 g_6 t_1^{-1} t_2^{-1}, g_5^2 = g_6 t_1, g_5^{g_1} = g_5 g_6, g_5^{g_2} = g_5 t_1^{-1}, g_5^{g_3} = g_5 t_1^{-1}, g_5^{g_4} = \\ &g_5 t_1^{-1}, g_6^2 = t_2, g_6^{g_1} = g_6 t_1^{-1} t_2^{-1}, g_6^{g_2} = g_6 t_1, g_6^{g_3} = g_6 t_2^{-1}, g_6^{g_4} = g_6 t_2^{-1}, g_6^{g_5} = g_6, t_1^{g_1} = \\ &t_1 t_2, t_1^{g_2} = t_1^{-1}, t_1^{g_3} = t_1^{-1}, t_1^{g_4} = t_1^{-1}, t_1^{g_5} = t_1, t_1^{g_6} = t_1, t_2^{g_1} = t_1^{-2} t_2^{-1}, t_2^{g_2} = t_1^2 t_2, t_2^{g_3} = \\ &t_2^{-1}, t_2^{g_4} = t_2^{-1}, t_2^{g_5} = t_2, t_2^{g_6} = t_2, t_2^{t_1} = t_2. \end{aligned}$$

$$\begin{aligned} \mathcal{W}_1 &= 2^5(\cdot, (0, 1), \cdot, (0, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)), \\ \mathcal{W}_2 &= 2^5(\cdot, (0, 0), \cdot, (1, 0), \cdot, (1, 0), (0, 0), \cdot, (0, 0), (0, 0), (1, 0), (1, 1), (1, 0), (1, 1), (1, 0), (0, 1), (1, 1), (0, 1), (0, 1)), \\ \mathcal{W}_3 &= 2^5(\cdot, (0, 0), \cdot, (0, 1), \cdot, (0, 1), (0, 0), \cdot, (0, 0), (0, 0), (0, 1), (0, 1), (0, 1), (0, 1), (0, 1), (0, 0), (0, 1), (0, 1), (0, 0), (0, 0)), \\ \mathcal{W}_4 &= 2^5(\cdot, (0, 0), \cdot, (0, 0), \cdot, (0, 0), (0, 1), \cdot, (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0)), \\ \mathcal{W}_5 &= 2^5(\cdot, (0, 0), \cdot, (0, 0), \cdot, (0, 0), (0, 0), \cdot, (0, 1), (0, 1), (0, 1), (0, 0), (0, 1), (0, 1), (0, 1), (0, 0), (0, 1), (0, 1), (0, 0), (0, 0)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_1 + \mathcal{W}_4, \mathcal{W}_1 + \mathcal{W}_3, \mathcal{W}_1 + \mathcal{W}_2, \mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4, \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_4, \mathcal{W}_3 + \mathcal{W}_4, \mathcal{W}_2 + \mathcal{W}_4\}. \end{aligned}$$

**$S_2$  with (6, 14).** Generators  $g_1, \dots, g_7, t_1, t_2$  and relations

$$\begin{aligned} g_1^2 &= g_4, g_2^2 = 1, g_2^{g_1} = g_2 g_3, g_3^2 = 1, g_3^{g_1} = g_3 g_5, g_3^{g_2} = g_3, g_4^2 = 1, g_4^{g_1} = g_4, g_4^{g_2} = \\ &g_4 g_5 g_7, g_4^{g_3} = g_4 g_6 g_7 t_2, g_5^2 = g_7 t_1^{-1}, g_5^{g_1} = g_5 g_6, g_5^{g_2} = g_5 g_7, g_5^{g_3} = g_5 g_7, g_5^{g_4} = \\ &g_5 g_7, g_6^2 = g_8 t_2^{-1}, g_6^{g_1} = g_6 g_7 t_2, g_6^{g_2} = g_6 g_7 t_1^{-1}, g_6^{g_3} = g_6 t_2, g_6^{g_4} = g_6 t_2, g_6^{g_5} = g_6, g_7^2 = \\ &t_1, g_7^{g_1} = g_7 t_2, g_7^{g_2} = g_7 t_1^{-1}, g_7^{g_3} = g_7 t_1^{-1}, g_7^{g_4} = g_7 t_1^{-1}, g_7^{g_5} = g_7, g_7^{g_6} = g_7, t_1^{g_1} = \\ &t_1 t_2^2, t_1^{g_2} = t_1^{-1}, t_1^{g_3} = t_1^{-1}, t_1^{g_4} = t_1^{-1}, t_1^{g_5} = t_1, t_1^{g_6} = t_1, t_2^{g_1} = t_1^{-1} t_2^{-1}, t_2^{g_2} = t_1 t_2, t_2^{g_3} = \\ &t_2^{-1}, t_2^{g_4} = t_2^{-1}, t_2^{g_5} = t_2, t_2^{g_6} = t_2, t_2^{t_1} = t_2. \end{aligned}$$

$$\begin{aligned} \mathcal{W}_1 &= 2^6(\cdot, \cdot, \cdot, (1, 0), \cdot, (1, 0), (0, 0), \cdot, \cdot, (0, 0), (1, 0), (1, 0), (1, 0), (1, 0), (0, 0), (1, 0), (1, 0), (0, 0), (0, 0), \cdot, \cdot, (0, 0)), \\ \mathcal{W}_2 &= 2^6(\cdot, \cdot, \cdot, (0, 0), \cdot, (0, 0), (1, 0), \cdot, \cdot, (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), \cdot, \cdot, (0, 0)), \\ \mathcal{W}_3 &= 2^6(\cdot, \cdot, \cdot, (0, 0), \cdot, (0, 0), (0, 0), \cdot, \cdot, (1, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (1, 0), (1, 0), (0, 0), (1, 0), \cdot, \cdot, (1, 0)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 + \mathcal{W}_2\}. \end{aligned}$$

**$S_3$  with (4, 6).** Generators  $g_1, \dots, g_5, t$  and relations

$$g_1^2 = g_4, g_2^2 = 1, g_2^{g_1} = g_2 g_3, g_3^2 = g_5, g_3^{g_1} = g_3 g_5 t^{-1}, g_3^{g_2} = g_3 g_5 t^{-1}, g_4^2 = 1, g_4^{g_1} = g_4, g_4^{g_2} = g_4, g_4^{g_3} = g_4, g_5^2 = t, g_5^{g_1} = g_5 t^{-1}, g_5^{g_2} = g_5 t^{-1}, g_5^{g_3} = g_5, g_5^{g_4} = g_5, t^{g_1} = t^{-1}, t^{g_2} = t^{-1}, t^{g_3} = t, t^{g_4} = t, t^{g_5} = t.$$

$$\begin{aligned} \mathcal{W}_1 &= 2^5((0), (1), (0), (0), (0), (0), (0), (0), (0), (0), (0), (0), (0), (0), (0)), \\ \mathcal{W}_2 &= 2^5((0), (0), (0), (0), (0), (1), (0), (0), (1), (0), (1), (1), (1), (0), (0)), \\ \mathcal{W}_3 &= 2^5((0), (0), (0), (0), (0), (0), (1), (0), (0), (0), (0), (0), (0), (0), (0)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3, \mathcal{W}_2 + \mathcal{W}_3\}. \end{aligned}$$

**$S_4$  with (3, 8).** Generators  $g_1, \dots, g_4, t$  and relations

$$g_1^2 = g_4, g_2^2 = g_3, g_2^{g_1} = g_2 g_3 t^{-1}, g_3^2 = t, g_3^{g_1} = g_3 t^{-1}, g_3^{g_2} = g_3, g_4^2 = 1, g_4^{g_1} = g_4, g_4^{g_2} = g_4, g_4^{g_3} = g_4, t^{g_1} = t^{-1}, t^{g_2} = t, t^{g_3} = t, t^{g_4} = t.$$

$$\begin{aligned} \mathcal{W}_1 &= 2^7((0), (0), (0), (0), (0), (0), (1), (0), (0), (0)), \\ \mathcal{W}_2 &= 2^6((0), (0), (0), (1), (1), (0), (-2), (0), (2), (0)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, 3\mathcal{W}_2, 2\mathcal{W}_2\}. \end{aligned}$$

**$S_5$  with (2, 6).** Generators  $g_1, \dots, g_3, t$  and relations

$$g_1^2 = 1, g_2^2 = t, g_2^{g_1} = g_2 t^{-1}, g_3^2 = 1, g_3^{g_1} = g_3, g_3^{g_2} = g_3, t^{g_1} = t^{-1}, t^{g_2} = t, t^{g_3} = t.$$

$$\begin{aligned} \mathcal{W}_1 &= 2^5((1), (0), (0), (0), (0), (0)), \\ \mathcal{W}_2 &= 2^5((0), (0), (1), (0), (0), (0)), \\ \mathcal{W}_3 &= 2^5((0), (0), (0), (0), (1), (0)), \\ \mathcal{W}_4 &= 2^5((0), (0), (0), (0), (0), (1)), \\ \mathcal{W} &\in \{0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_1 + \mathcal{W}_4\}. \end{aligned}$$

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